Practice Midterm 1

1. Systems A and B consist of four linear equations in four unknowns modulo 2.

$$\begin{array}{|ccccc|}\hline x_1 & + & x_2 & = & 0 \\ + & & + & & \\ y_1 & + & y_2 & = & 0 \\ = & & = & & \\ 1 & & 1 & & \\ \hline \end{array} \qquad \begin{array}{|ccccc|}\hline x_1 & + & x_2 & = & 1 \\ + & & + & & \\ y_1 & + & y_2 & = & 0 \\ = & & = & & \\ 1 & & 1 & & \\ \hline \end{array}$$

system A system B

   (a) Solve system A using modular Gauss-Jordan elimination. If there are multiple solutions, specify the free variables and the assignment to the remaining variables in terms of the free variables.

   (b) In system B the value in the first equation was flipped. Produce a contradictory linear combination for system B. You may use any method you like.

2. You want to apply Gradient Descent to the system

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   The eigenvalues of the matrix are $\lambda_1 = 4$ and $\lambda_2 = -2$ with corresponding eigenvectors $\mathbf{v}_1 = (1/\sqrt{2}, 1/\sqrt{2})$, $\mathbf{v}_2 = (1/\sqrt{2}, -1/\sqrt{2})$.

   (a) What is the maximum rate ρ^* below which convergence to the unique solution $(1/4, 1/4)$ is guaranteed (for any initialisation)?

   (b) You run Gradient Descent with rate $\rho = 0.01$ and initialisation $(x, y) = (1, 0)$. The distance between the state (x_t, y_t) at time t and the solution $(1/4, 1/4)$ is $\Theta(b^t)$ for some number b. Find b.

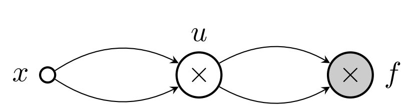
3. The list representation of f defined over the fourth roots of unity is

x	1	i	-1	$-i$
$f(x)$	0	1	0	1

   (a) Find the polynomial representation of f. You may use any method you like. Explain your steps.

   (b) What is the linear approximation of f that minimizes the average square error?

4. Show the result of applying backpropagation to the circuit below. Your circuit may use plus and times gates. If you applied any simplifications (e.g., $1 \times x$ was replaced by x) explain them.



Practice Midterm 2

1. Apply Gauss-Jordan Elimination to find a contradictory linear combination for the system of equations below. Explain all the steps.

$$x - y = 1$$
$$y - z = 1$$
$$z - x = 1$$

2. You apply Power Iteration on symmetric matrix A with initialization $\mathbf{x}$.

   (a) Let $\mathbf{x}$ be the state at the end of step t and $\mathbf{y}$ be the state in step $t+1$ before normalization. Prove that the spectral norm of A is at least $\|\mathbf{y}\|/\|\mathbf{x}\|$, regardless of initialization.

   (b) Use part (a) to argue that the following matrix has spectral norm greater than 3.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

3. The function $f(x, y, z)$ evaluates to 1 if all its inputs are $+1$, and to -1 if at least one of them is a -1. Apply the Fast Fourier-Walsh algorithm to calculate the Fourier coefficients of f. Show all the steps. You may shortcut the execution if an intermediate function simplifies to a constant.

4. In this question all times gates take exactly two inputs. Assume n is a power of two.

   (a) Draw a circuit for x^n with $\log n$ times gates.

   (b) Let $f(x) = 1 + x + x^2 + \cdots + x^{n-1}$. Show that $f(x) = (1 + x)(1 + x^2)(1 + x^4) \cdots (1 + x^{n/2})$.
   (**Hint:** Induction.)

   (c) Use part (b) to argue that f has a circuit (with plus and times gates) of size $O((\log n)^2)$.

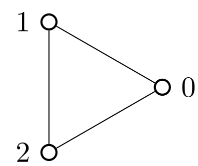
   (d) Let $g(x) = 1 + 2x + 3x^2 + \cdots + (n - 1)x^{n-2}$. Show that g has a circuit of size $O((\log n)^2)$.
   (**Hint:** Backpropagation.)

Practice Midterm 3

1. Apply Gauss-Jordan Elimination to find a *nonzero* solution to the linear system below. Explain your steps.

$$\begin{aligned} x + y + z &= 0 \\ x + 2y + 4z &= 0 \end{aligned}$$

2. Let A be the adjacency matrix of the 3-cycle (with zeros on the diagonal). The largest eigenvalue of A is $\lambda_1 = 2$. The corresponding eigenvector is $\mathbf{v}_1 = (1/\sqrt{3})(1, 1, 1)$.



   (a) Let $\mathbf{x}$ be any vector that is orthogonal to $\mathbf{v}_1$. Show that the state of Power Iteration on A initialized with $\mathbf{x}$ oscillates between $\mathbf{x}$ and $-\mathbf{x}$. Assume there are no precision errors.

   (b) Use part (a) to deduce the other two eigenvalues λ_2 and λ_3 of A. What are they?

3. Calculate the Fourier transforms of the following functions. You may use any method you like. Write your answer in the tables provided. $+$ and $-$ are shorthand for the numbers $+1$ and -1.

   (a)

x_1x_2	$++$	$-+$	$+-$	$--$
$f(x_1x_2)$	1	1	-1	-1

S	$\varnothing$	1	2	12
$\hat{f}(S)$				

   (b)

x_1x_2	$++$	$-+$	$+-$	$--$
$g(x_1x_2)$	0	1	1	1

S	$\varnothing$	1	2	12
$\hat{g}(S)$				

   (**Hint:** What is $1 - g$?)

   (c) $h(x_1x_2x_3) = \begin{cases} f(x_1x_2), & \text{if } x_3 = -1 \\ g(x_1x_2), & \text{if } x_3 = +1. \end{cases}$

S	$\varnothing$	1	2	3	12	13	23	123
$\hat{g}(S)$								

   (**Hint:** Apply one of the steps of the Fast Fourier-Walsh algorithm.)

4. Show the result of forward propagation on the circuit $f(x) = x(1 + x(1 + x))$. Your circuit may use plus and times gates. If you applied any simplifications explain them.