Practice questions

1. Cup 1 has 3 blue balls and 1 red ball. Cup 2 has 1 blue ball and 3 red balls. You choose a cup at random and draw three balls from it without replacement. Let C_i and B_j be the events "you chose cup i" and "the j-th ball is blue", respectively. Is it true that:

   (a) B_1 and B_2 are independent given C_1.

   (b) B_1 and B_2 are independent.

   (c) B_2 and B_3 are independent given B_1.

2. Suppose the number of school bus arriving at the Sir Run Run Shaw Hall in any time interval is a Poisson random variable, with a rate of 1 bus in 5 minutes.

   (a) What is the probability that no bus arrives in the next 30 minutes?

   (b) What is the probability that at least 5 buses arrive in the next 10 minutes?

   (c) Let M be the minute in which the first bus arrives. For example if the first bus arrives in 4 min 23 sec then $M = 5$. What kind of random variable is M? What is $\mathrm{P}(M = 6)$?

3. You attend a wedding with 500 guests.

   (a) Let X be the number of other guests that share your birthday. What kind of random variable is X?

   (b) What is the probability that $X = 1$?

   (c) Now model the number of other guests that share your birthday as a Poisson(λ) random variable N. What is the rate λ? How do $\mathrm{P}(X = 1)$ and $\mathrm{P}(N = 1)$ compare?

4. Calculate the PMFs of the following random variables:

   (a) The first time X at which both the patterns `TH` and `HT` have appeared in a sequence of fair coin flips. For example, $X = 6$ for the sequence `HHTTTH`.

   (b) The first time Y at which all three face values have appeared in a sequence of rolls of a fair 3-sided die. For example, $Y = 6$ for the sequence `232231`.

Additional ESTR 2018 questions

5. The Poisson random variable measures the number of successes in many *independent* trials, each with small success probability. In some cases, the Poisson random variable gives an approximate answer even if the trials are not completely independent. It is not so easy to explain the general conditions under which such “Poisson approximation” works but here is one example.

   In Lecture 1 we talked about a system with n antennas on a line, where each of the antennas can be functional or defective. The system fails whenever two *consecutive* antennas are defective. Assume the defects are independent and each happens with probability p.

   (a) Let P_n be the probability that the system fails. Derive a recurrence for P_n. Solve this recurrence for $n = 10$ and $p = 0.1$.

   (b) Say a *failure* happens at position i if the i-th and $(i + 1)$-st antenna are both defective. Argue that the events ”A failure happens at position i”, $i = 1, \ldots, n - 1$ are not independent. What is the probability q of failure at position i?

   (c) Let N be a Poisson($(n - 1)q$) random variable. Calculate $P(N \neq 0)$ and compare to part (a).

6. The *hot hand paradox* is the belief that if your favorite sports team is on a “winning streak” then it is more likely to win the next game. For example, in this sequence of 38 wins and losses

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   LLLWWWLWLLLLWWLLLLLWLLWWLWWLWLLWLLLWWLWLLLWLLWWWWWWWWWWWWLLLLW
   ```

   there are 12 consecutive wins. Was the team on a winning streak?

   The Premier League has 20 football teams that play 38 rounds of matches. Let S be the length of the longest winning streak in the league. Come up with a probability model and estimate the probability mass function and the expected value of S. I suggest that you start with a simple model that you can simulate (for instance, disregard draws), and then move to an even simpler one that you can analyze.