Questions

1. A Chicken lays a Poisson(λ) number N of eggs. Each egg independently hatches a chick with probability p. Let X be the number of chicks that hatch. Calculate
   (a) the conditional expectation $\mathrm{E}[X|N = n]$;
   (b) the unconditional expectation $\mathrm{E}[X]$;
   (c) the unconditional expectation $\mathrm{E}[NX]$;
   (d) the covariance $\mathrm{Cov}[X, N]$.

   [Based on Blitzstein-Hwang Exercise 7.48]

2. Six boys and six girls sit randomly at a round table. Let X be the number of boys that sit between two girls.
   (a) Let X_i be the indicator for the event "boy i sits between two girls." What is $\mathrm{Var}[X_i]$?
   (b) What is $\mathrm{Cov}[X_i, X_j]$ ($i \neq j$)?
   (c) What is $\mathrm{Var}[X]$?

3. Two typing monkeys sit at special keyboards. The keyboards have only two keys `a` and `b`. Each monkey types in a random 200 letter string, independently of the other one. Let E be the event "There is a pattern of 20 consecutive letters that appears in both strings." Is it true that $\mathrm{P}(E) < 5\%$? Justify your answer.

4. 100 people put their hats in a box and each one pulls a random hat out.
   (a) Let G be any 10-person group. What is the probability that everyone in G pulls their own hat?
   (b) What is the expected *number* of 10-person groups in which everyone pulls their own hat?
   (c) Show that the probability that 10 or more people pull their own hat is less than 10^{-6}.

Additional ESTR 2018 question

5. In ESTR 2018 Lecture 9 I claimed that $\mathrm{E}[X^4] = 3\sigma^4$ for a Normal$(0, \sigma)$ random variable X. In this exercise you will derive that formula.
   (a) Let X be a Normal(0, 1) random variable. Show that $\mathrm{E}[e^{tX}] = e^{-t^2/2}$ for every real number t.
   (b) Calculate $\mathrm{E}[X^4]$ by taking derivatives. (You may assume that the expectation of a derivative is the same as the derivative of an expectation.)
   (c) Can you calculate $\mathrm{E}[X^k]$ for all k?