1. You choose a 3-digit number by randomly setting its digits to `1`, `2`, `3`, `4` or `5` without repetition. What is the probability that your number is greater than `232`?

   Solution: By the equally likely outcomes formula the probability is $|A|/5 \cdot 4 \cdot 3$ where A is the set of numbers of the given type that are greater than `232`. These comprise all $3 \cdot 4 \cdot 3 = 36$ choices that start with `3`, `4`, or `5`, all $2 \cdot 3 = 6$ choices that start with a `2` followed by a `4` or `5`, and 2 choices that start with `23` followed by a `4` or a `5`. Therefore $|A| = 24 + 6 + 2 = 44$ and $\mathrm{P}(A) = 44/60 = 11/15$.

   Alternative solution: Let X, Y, Z be the three digits. The event A of interest is a disjoint union of $X \geq 3$, $(X = 2) \cap (Y \geq 4)$, and $(X = 2) \cap (Y = 3) \cap (Z \geq 4)$. By the axioms of probability

$$\begin{aligned}\mathrm{P}(A) &= \mathrm{P}(X \geq 3) + \mathrm{P}(Y \geq 4 | X = 2)\,\mathrm{P}(X = 2) + \mathrm{P}(Z \geq 4 | X = 2, Y = 3)\,\mathrm{P}(Y = 3 | X = 2)\,\mathrm{P}(X = 2) \\ &= \frac{3}{5} + \frac{1}{2} \cdot \frac{1}{5} + \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} = \frac{11}{15}.\end{aligned}$$

2. A river is polluted with probability 40%. Its fish stocks are low with probability 50%. These events are independent. Fishing is forbidden when the river is polluted or fish stocks are low. You see a "fishing is forbidden" sign. What is the probability that the river is polluted?

   Solution: Let A and B be the two events. Then

$$\mathrm{P}(A|A \cup B) = \frac{\mathrm{P}(A)}{\mathrm{P}(A \cup B)} = \frac{\mathrm{P}(A)}{1 - \mathrm{P}(A^c)\,\mathrm{P}(B^c)} = \frac{0.4}{1 - 0.6 \cdot 0.5} = \frac{4}{7} \approx 0.571.$$

3. Alice, Bob, and Charlie each put \$1 on the table and flip a fair coin. The one whose flip is different from the other two wins and collects the money. If all flips are identical they keep flipping until a winner emerges, putting in an extra \$1 each time. What is the expected gain of the winner?

   Solution: If a winner emerges in round R their gain is $2R$ dollars. In any given round a winner emerges in six outcomes out of the eight possible. As outcomes are equally likely the probability is $3/4$. Therefore R is a Geometric(3/4) random variable. Its expected value is $\mathrm{E}[R] = 4/3$ and the winner's gain is $\mathrm{E}[2R] = 2\,\mathrm{E}[R] = 8/3$.

4. A cup has 9 balls of which 2 are blue, 3 are red, and 4 are green. Three balls are taken out without replacement. Find the PMF of the number of colors that appear among the balls taken out.

   Solution: The number of colors N can be 1, 2, or 3. The event $N = 1$ happens when all three balls are red or green, so $\mathrm{P}(N = 1) = \frac{3}{9} \cdot \frac{2}{8} \cdot \frac{1}{7} + \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} = 5/84$. For the event $N = 3$, three balls of different colors in some fixed order, e.g. blue then red then green, has probability $2 \cdot 3 \cdot 4/9 \cdot 8 \cdot 7 = 1/21$. As all $3! = 6$ color orderings are equally likely $\mathrm{P}(N = 3) = 6/21$. Since the probabilities must add up to one $\mathrm{P}(N = 2) = 1 - 5/84 - 6/21 = 55/84$.

5. Six boys and six girls sit at a round table. What is the expected number of boys that sit between two girls?

   Solution: The probability that a given boy has a girl to his left is $6/11$. Conditioned on this, the probability that he has a girl to his right is $5/10 = 1/2$. Therefore the probability that a given boy sits between two girls is $3/11$. By linearity of expectation the number of boys that sit between two girls is $6 \cdot 3/11 = 18/11 \approx 1.636$.