Practice Final 1

1. The joint probability density function of the lifetimes X and Y of two connected components in a machine is

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x(1+y)}, & x \geq 0, y \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the probability that the lifetime X of the first component exceeds 3?

   (b) Are X and Y independent? Justify your answer.

2. Alice sends a message a that equals -1 or 1. Bob receives the value B which is a Normal random variable with mean a and standard deviation 0.5. Bob guesses that Alice sent 1 if $B > 0.5$, that Alice sent -1 if $B < -0.5$, and declares failure otherwise (when $|B| \leq 0.5$).

   (a) What is the probability that Bob declares failure?

   (b) Given that Bob didn't declare failure, what is the probability that his guess is correct?

3. The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. There are 20 floors above (not including) the ground floor and each person is equally likely to get off on any one of them, independently of all others.

   (a) What is the probability p that the elevator doesn't stop on the seventh floor?

   (b) What is the expected number of stops that the elevator will make?
   (Express the answer in terms of p in case you didn't complete part (a).)

4. Alice takes T hours to travel to Bob's house, where T is a random variable with PDF

$$f_T(t) = \begin{cases} 1/t^2, & \text{when } t \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the CDF (cumulative distribution function) $F_T(t) = \mathrm{P}(T \leq t)$.

   (b) The distance between Alice's and Bob's house is one mile so that Alice travels at a speed $V = 1/T$ miles per hour. What is Alice's expected speed $\mathrm{E}[V]$?

5. n independent random numbers are sampled uniformly from the interval $[0, 1]$.

   (a) If $n = 10$, what is the probability that exactly 4 of them are greater than 0.7?

   (b) If $n = 50$, use the Central Limit Theorem to estimate the probability that their sum is between 20 and 25 (inclusive).

6. A group of 10 boys and 10 girls is randomly divided into 5 teams A, B, C, D, E with 4 children per team.

   (a) What is the probability that all children in team A are of the same gender?

   (b) Is the probability that all teams are of mixed gender more than 50% or not? Justify your answer.

Practice Final 2

1. Let A and B be arbitrary events. Which of the following is true? If you answer yes, prove it using the axioms of probability. If you answer no, provide a counterexample.

   (a) $\mathrm{P}(A|B) + \mathrm{P}(A|B^c) = 1$.

   (b) $\mathrm{P}(A \cap B | A \cup B) \leq \mathrm{P}(A|B)$.

2. X and Y are independent random variables, both with the following PMF:

x	1	2	4
$f(x)$	1/3	1/3	1/3

   (a) Find the PMF of $X + Y$.

   (b) Are X and $X + Y$ independent? Justify your answer.

3. The number of cars behind a traffic light at the time it turns green is a Poisson random variable X with mean 1. The number of cars that cross the green light is $\min\{X, 3\}$.

   (a) Find the PMF of the number of cars that cross the (green) light.

   (b) The light turns green 50 times within the hour. Is the probability that more than 100 cars cross within the hour larger or smaller than 50%? Justify your answer.

4. Alice and Bob independently arrive at the bus stop at a uniformly random time between 8 and 9. There are buses at 8.15, 8.30, and 9.

   (a) What is the probability that they catch the same bus?

   (b) Given that Bob didn't run into Alice on the 8.30 bus, what is the probability that Alice caught the 8.15 bus?

5. The body weight of a random person is a normal random variable with mean 60kg and standard deviation 10kg. The carrying capacity of an elevator is 600kg. If nine people enter the elevator, what is the probability that the weight limit is exceeded? Assume their weights are independent.

6. A deck of cards is divided into 26 pairs. Let X be the number of those pairs in which both cards are of the same suit. (A deck of cards has 4 suits and each suit has 13 cards.)

   (a) What is the expected value of X?

   (b) What is the variance of X?

Practice Final 3

1. Urn A has 4 blue balls. Urn B has 1 blue ball and 3 red balls.

   (a) You draw a ball from a random urn and it is blue. What is the probability that it came from urn A?

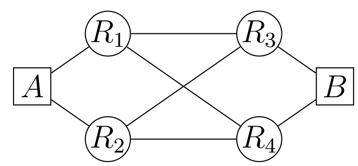
   (b) You draw another ball from the same urn. What is the probability that the second ball is also blue?

2. A radio station gives a gift to the third caller who knows the birthday of the radio talk show host. Each caller has a 0.7 probability of guessing the host's birthday, independently of other callers.

   (a) What is the probability mass function of the number of calls necessary to find the winner?

   (b) What is the probability that the station will need five or more calls to find a winner?

3. Computers A and B are linked through routers R_1 to R_4 as in the picture. Each router fails independently with probability 10%.



   (a) What is the probability there is a connection between A and B?

   (b) Are the events "there is a connection between A and B" and "exactly two routers fail" independent? Justify your answer.

4. A bus takes you from A to B in 10 minutes. On average a bus comes once every 5 minutes. A taxi takes you in 5 minutes, and on average a taxi comes once every 10 minutes. Their arrival times are independent exponential random variables. A bus comes first.

   (a) If you want to minimize the (expected) travel time, should you take this bus?

   (b) If you do take the bus, what is the probability that you made the wrong decision?

5. 10 people toss their hats and each person randomly picks one. The experiment is repeated one more time.

   (a) What is the probability that Bob picked his own hat both times?

   (b) Let A be the event that at least one person picked their own hat both times. True or false: $P(A) > 25\%$? Justify your answer.

6. You are offered to play the following game for \$2: Toss three 3-sided dice and collect D dollars, where D is the number of distinct face values that appear.

   (a) Is it rational for you to play this game? Justify your answer.

   (b) Apply the Central Limit Theorem to estimate the probability that you come out ahead (you earn some money) after playing 90 rounds of the game.

Random Variable X	PMF/PDF $f_X(x)$	E[X]	Var[X]
Binomial(n, p)	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Geometric(p)	$(1-p)^{x-1} p$	$1/p$	$(1-p)/p^2$
Poisson(λ)	$\lambda^x e^{-\lambda}/x!$	λ	λ^2
Exponential(λ)	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
Normal(μ, σ^2)	$(2\pi\sigma^2)^{-1/2} \cdot e^{-(x-\mu)^2/2\sigma^2}$	μ	σ^2

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Normal Cumulative Distribution Function $P(Z \leq z)$