Practice Final 1

1. The joint probability density function of the lifetimes X and Y of two connected components in a machine is

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x(1+y)}, & x \geq 0, y \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the probability that the lifetime X of the first component exceeds 3?

   Solution: $\mathrm{P}(X > 3) = \int_3^\infty \int_0^\infty xe^{-x(1+y)}dydx = \int_3^\infty e^{-x}dx = e^{-x}|_3^\infty = 0.05.$

   (b) Are X and Y independent? Justify your answer.

   Solution: No. If X and Y were independent then

$$f_{X,Y}(1,1)f_{X,Y}(2,2) = 2e^{-8} \quad \text{and} \quad f_{X,Y}(1,2)f_{X,Y}(2,1) = 2e^{-7}$$

   should both be equal to the same value $f_X(1)f_X(2)f_Y(1)f_Y(2)$.
   Alternatively one can calculate the marginal PDFs

$$f_X(x) = \int_0^\infty xe^{-x(1+y)}dy = e^{-x},$$
$$f_Y(y) = \int_0^\infty xe^{-x(1+y)}dx = \frac{1}{(1+y)^2}$$

   for $x, y \geq 0$ and conclude that $f_X(x)f_Y(y)$ is not the same function as $f_{X,Y}(x,y)$.

2. Alice sends a message a that equals -1 or 1. Bob receives the value B which is a Normal random variable with mean a and standard deviation 0.5. Bob guesses that Alice sent 1 if $B > 0.5$, that Alice sent -1 if $B < -0.5$, and declares failure otherwise (when $|B| \leq 0.5$).

   (a) What is the probability that Bob declares failure?

   Solution: For either message, failure occurs when a normal random variable is between 1 and 3 standard deviations from the mean on one side. If N is a Normal$(0,1)$ random variable then

$$\mathrm{P}(1 \leq N < 3) = \mathrm{P}(N < 3) - \mathrm{P}(N < 1) \approx 0.9987 - 0.8413 = 0.1574.$$

   (b) Given that Bob didn't declare failure, what is the probability that his guess is correct?

   Solution: By symmetry we may assume Alice sent 1. The event "Bob's guess is correct *and* failure didn't occur" happens when B takes value 0.5 or larger, or when a Normal$(0,1)$ random variable N takes value at most 1, which is approximately 0.8413. Therefore the conditional probability that Bob's guess is correct is about $0.8413/(1 - 0.1574) \approx 0.9985$.

3. The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. There are 20 floors above (not including) the ground floor and each person is equally likely to get off on any one of them, independently of all others.

   (a) What is the probability p that the elevator doesn't stop on the seventh floor?

   Solution: The number of passengers bound for the seventh floor is a Poisson random variable with mean $10/20 = 1/2$, so the probability that no passengers land there is the probability that this random variable takes value zero, which is $p = e^{-1/2} \approx 0.6065$.

   Alternatively, by conditioning on the number N of passengers who enter the elevator and applying the total probability theorem,

$$p = \sum_{n=0}^{\infty} (19/20)^n \cdot \mathrm{P}(N = n) = \sum_{n=0}^{\infty} (19/20)^n \cdot e^{-10} \cdot \frac{10^n}{n!}$$

$$= e^{-10} \cdot \sum_{n=0}^{\infty} \frac{(10 \cdot 19/20)^n}{n!} = e^{-10} \cdot e^{10 \cdot 19/20} = e^{-1/2}.$$

   (b) What is the expected number of stops that the elevator will make?
   (Express the answer in terms of p in case you didn't complete part (a).)

   Solution: By linearity of expectation, the expected number of stops is the sum of the probabilities that the elevator stops on floor 1 up to floor 20. By part (a) each of these probabilities is $1 - e^{-1/2}$ so the answer is $20 \cdot (1 - e^{-1/2}) \approx 7.869$.

4. Alice takes T hours to travel to Bob's house, where T is a random variable with PDF

$$f_T(t) = \begin{cases} 1/t^2, & \text{when } t \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the CDF (cumulative distribution function) $F_T(t) = \mathrm{P}(T \leq t)$.

   Solution: $F_T(t)$ is zero when $t < 1$. When $t \geq 1$,

$$F_T(t) = \int_1^t \frac{1}{u^2}\, du = -\frac{1}{u}\bigg|_1^t = 1 - \frac{1}{t}.$$

   (b) The distance between Alice's and Bob's house is one mile so that Alice travels at a speed $V = 1/T$ miles per hour. What is Alice's expected speed $\mathrm{E}[V]$?

   Solution: The CDF $F_V(v)$ of V is zero when $v \leq 0$. If $v > 0$,

$$\mathrm{P}(V \leq v) = \mathrm{P}(1/T \leq v) = \mathrm{P}(T \geq 1/v) = 1 - F_T(1/v) = \begin{cases} v, & \text{if } 0 \leq v \leq 1, \\ 1, & \text{if } v \geq 1. \end{cases}$$

   This is the CDF of a Uniform$(0, 1)$ random variable, so $\mathrm{E}[V] = 1/2$.

5. n independent random numbers are sampled uniformly from the interval $[0, 1]$.

(a) If $n = 10$, what is the probability that exactly 4 of them are greater than 0.7?

Solution: Let N be the number of such random numbers greater than 0.7. Then N is a binomial random variable with $n = 10$ samples and success probability $p = 3$, so

$$\mathrm{P}(X = 4) = \binom{10}{4} 0.3^4 (1 - 0.3)^{10-4} \approx 0.200.$$

(b) If $n = 50$, use the Central Limit Theorem to estimate the probability that their sum is between 20 and 25 (inclusive).

Solution: Let X_i denote the value of the i-th random number ($i = 1, \dots, n$). Then $X_1, \dots, X_n$ are independent random variables with mean $1/2$ and variance $1/12$. Let $X = X_1 + \cdots + X_n$. Then $\mathrm{E}[X] = 25$ and $\mathrm{Var}[X] = 50/12$. By the Central Limit Theorem, the CDF of X can be approximated by the CDF of a Normal$(25, 50/12)$ random variable $\tilde{N}$. Normalizing $\tilde{N} = 25 + N \cdot \sqrt{50/12}$,

$$\begin{aligned}
\mathrm{P}(20 \le X \le 25) &\approx \mathrm{P}(20 \le \tilde{N} \le 25) \\
&= \mathrm{P}(20 \le 25 + N \cdot \sqrt{50/12} \le 25) \\
&= \mathrm{P}(-5/\sqrt{50/12} \le N \le 0) \\
&\approx \mathrm{P}(-2.45 \le N \le 0) \\
&= F_N(0) - F_N(-2.45) \\
&\approx 0.5 - 0.0071 \\
&\approx 0.4929.
\end{aligned}$$

6. A group of 10 boys and 10 girls is randomly divided into 5 teams A, B, C, D, E with 4 children per team.

(a) What is the probability that all children in team A are of the same gender?

Solution: By the multiplication rule, this probability is $p = 9/19 \cdot 8/18 \cdot 7/17 \approx 0.087$.

(b) Is the probability that all teams are of mixed gender more than 50% or not? Justify your answer.

Solution: It is. Let S be the number of same gender teams. By linearity of expectation, $\mathrm{E}[S] = 5p \approx 0.433$. By Markov's inequality, $\mathrm{P}(S \ge 1) \le \mathrm{E}[S]$, so the complementary event $S = 0$ occurs with probability at least $1 - 0.433 > 0.5$.

Practice Final 2

1. Let A and B be arbitrary events. Which of the following is true? If you answer yes, prove it using the axioms of probability. If you answer no, provide a counterexample.

(a) $\mathrm{P}(A|B) + \mathrm{P}(A|B^c) = 1$.

Solution: No. If B is the event of a fair coin flipping heads and A is the event of the coin flipping heads or tails then $\mathrm{P}(A|B) = 1$ and $\mathrm{P}(A|B^c) = 1$.

(b) $\mathrm{P}(A \cap B | A \cup B) \leq \mathrm{P}(A|B)$.

Solution: Yes, because $\mathrm{P}(A \cup B) \geq \mathrm{P}(B)$ and so

$$\mathrm{P}(A \cap B | A \cup B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A \cup B)} \leq \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \mathrm{P}(A|B).$$

2. X and Y are independent random variables, both with the following PMF:

x	1	2	4
$f(x)$	1/3	1/3	1/3

(a) Find the PMF of $X + Y$.

Solution: Let $Z = X + Y$. Using the convolution formula $\mathrm{P}(Z = z) = \sum_x \mathrm{P}(X = x)\,\mathrm{P}(Y = z - x)$ we obtain the following PMF:

z	2	3	4	5	6	8
$f_Z(z)$	1/9	2/9	1/9	2/9	2/9	1/9

(b) Are X and $X + Y$ independent? Justify your answer.

Solution: No, because $\mathrm{P}(X = 1, X + Y = 3) = \mathrm{P}(X = 1, Y = 2) = \mathrm{P}(X = 1)\,\mathrm{P}(Y = 2) = 1/9$, while $\mathrm{P}(X = 1)\,\mathrm{P}(X + Y = 3) = 1/3 \cdot 2/3 = 2/9$.

3. The number of cars behind a traffic light at the time it turns green is a Poisson random variable X with mean 1. The number of cars that cross the green light is $\min\{X, 3\}$.

(a) Find the PMF of the number of cars that cross the (green) light.

Solution: Let Y be this number. Then Y and X have the same probability of taking values 0, 1, and 2. Since probabilities must add up to one the event $Y = 3$ must be assigned the remaining probability. Using the Poisson PMF formula $\mathrm{P}(X = x) = 1/(x!e)$ we obtain the following PDF for Y:

y	0	1	2	3
$f(y)$	$1/e$	$1/e$	$1/2e$	$1 - 5/2e$.

(b) The light turns green 50 times within the hour. Is the probability that more than 100 cars cross within the hour larger or smaller than 50%? Justify your answer.

Solution: The expected number of cars that cross a green light is

$$\mathrm{E}[Y] = 1 \cdot \frac{1}{e} + 2 \cdot \frac{2}{e} + 3\left(1 - \frac{5}{2e}\right) = 3 - \frac{11}{2e} \approx 0.977.$$

By linearity of expectation, the expected number of cars that cross within the hour is about $50 \cdot 0.977$. By Markov's inequality, the probability that more than $2 \cdot 50 \cdot 0.977 = 97.7$ cars cross within the hour is less than 50%.

(Alternatively you can apply Markov's inequality to the sum of the X's and use the fact that the Y's are never larger than the X's.)

4. Alice and Bob independently arrive at the bus stop at a uniformly random time between 8 and 9. There are buses at 8.15, 8.30, and 9.

(a) What is the probability that they catch the same bus?

Solution: We model A and B as independent Uniform$(0,1)$ random variable representing the fraction of the hour at which Alice show up. The event E of interest is "$A, B \leq 1/4$ or $1/4 < A, B \leq 1/2$ or $A, B > 1/2$". Since the events are disjoint and A, B are independent,

$$\begin{aligned} \mathrm{P}(E) &= \mathrm{P}(A, B \leq 1/4) + \mathrm{P}(1/4 < A, B \leq 1/2) + \mathrm{P}(A, B > 1/2) \\ &= \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}. \end{aligned}$$

(b) Given that Bob did't run into Alice on the 8.30 bus, what is the probability that Alice caught the 8.15 bus?

Solution: Let F be the event that Alice did not arrive between 8.15 and 8.30, namely "$A \leq 1/4$ or $A > 1/2$". Then

$$\begin{aligned} \mathrm{P}(A \leq 1/4 | F) &= \frac{\mathrm{P}(A \leq 1/4 \text{ and } F)}{\mathrm{P}(F)} = \frac{\mathrm{P}(A \leq 1/4)}{\mathrm{P}(A \leq 1/4) + \mathrm{P}(A > 1/2)} \\ &= \frac{1/4}{1/4 + 1/2} = \frac{1}{3}. \end{aligned}$$

5. The body weight of a random person is a normal random variable with mean 60kg and standard deviation 10kg. The carrying capacity of an elevator is 600kg. If nine people enter the elevator, what is the probability that the weight limit is exceeded? Assume their weights are independent.

Solution: The weight of all nine people is a sum of nine independent Normal$(60, 10)$ random variables, which is a normal random variable of mean $9 \cdot 60 = 540$ and standard deviation $\sqrt{9} \cdot 10 = 30$. To exceed the carrying capacity the weight has to exceed its mean by more than two standard deviations. From the table this probability is approximately $1 - 0.9772 = 0.0228$, or 2.28%.

6. A deck of cards is divided into 26 pairs. Let X be the number of those pairs in which both cards are of the same suit. (A deck of cards has 4 suits and each suit has 13 cards.)

(a) What is the expected value of X?

Solution: We can write $X = X_1 + \cdots + X_{26}$ where X_i is 1 if the cards in the i-th pair are of the same suit and 0 if not. Then $\mathrm{E}[X_i] = \mathrm{P}(X_i = 1)$ is the probability that the i-th pair's cards are of the same suit, which is 12/51 because conditioned on the first card's suit, there are 12 out of 51 identical choices for the second one. By linearity of expectation $\mathrm{E}[X] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_{26}] = 26 \cdot 12/51 \approx 6.118$.

(b) What is the variance of X?

Solution: The variance of X is the sum of the 26 variances of X_i and the $26 \cdot 25$ covariances of X_i and X_j. The variance of X_i is $v = (12/51) \cdot (1 - 12/51) \approx 0.1799$. The covariance of X_i and X_j is

$$\mathrm{E}[X_i X_j] - \mathrm{E}[X_i]\,\mathrm{E}[X_j] = \mathrm{P}(X_i = 1, X_j = 1) - \mathrm{P}(X_i = 1)\,\mathrm{P}(X_j = 1).$$

The term $\mathrm{P}(X_i = 1, X_j = 1)$ is the probability of the event A that within both the i-th pair and the j-th pair, both cards are of the same suit. We can calculate this using the total probability theorem applied to the event E that the first card of the i-th pair and the first card of the j-th pair are of the same suit:

$$\mathrm{P}(X_i = 1, X_j = 1) = \mathrm{P}(A) = \mathrm{P}(A|E)\,\mathrm{P}(E) + \mathrm{P}(A|E^c)\,\mathrm{P}(E^c).$$

The probability of E is $12/51$. Conditioned on E, A happens if the second cards of both pairs are also of the same suit, which is $11/50 \cdot 10/49$. Conditioned on E^c—for example, if the i-th pair's first card is a heart and the j-th pair first card is a spade—A happens if the second cards are a heart and a spade respectively, which happens with probability $(12/50) \cdot (12/49)$, and so

$$\mathrm{P}(A) = \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{12}{51} + \frac{12}{50} \cdot \frac{12}{49} \cdot \left(1 - \frac{12}{51}\right).$$

Therefore the covariance of X_i and X_j equals

$$c = \mathrm{P}(A) - \left(\frac{12}{51}\right)^2 \approx 0.0001469.$$

Finally, $\mathrm{Var}[X] = 26 \cdot v + 26 \cdot 25 \cdot c \approx 4.7737$.

Practice Final 3

1. Urn A has 4 blue balls. Urn B has 1 blue ball and 3 red balls.

   (a) You draw a ball from a random urn and it is blue. What is the probability that it came from urn A?

   Solution: Let B_1 be the event the ball is blue and A be the event the ball came from urn A. By Bayes' rule

   $$\mathrm{P}(A|B_1) = \frac{\mathrm{P}(B_1|A)\,\mathrm{P}(A)}{\mathrm{P}(B_1|A)\,\mathrm{P}(A) + \mathrm{P}(B_1|A^c)\,\mathrm{P}(A^c)} = \frac{1 \cdot (1/2)}{1 \cdot (1/2) + (1/4) \cdot (1/2)} = \frac{4}{5}.$$

   (b) You draw another ball from the same urn. What is the probability that the second ball is also blue?

   Solution: Let B_2 be the event that the second ball is blue. By the total probability theorem

   $$\mathrm{P}(B_2|B_1) = \mathrm{P}(B_2|B_1, A)\,\mathrm{P}(A|B_1) + \mathrm{P}(B_2|B_1, A^c)\,\mathrm{P}(A^c|B_1) = 1 \cdot \frac{4}{5} + 0 \cdot \left(1 - \frac{4}{5}\right) = \frac{4}{5}.$$

2. A radio station gives a gift to the third caller who knows the birthday of the radio talk show host. Each caller has a 0.7 probability of guessing the host's birthday, independently of other callers.

   (a) What is the probability mass function of the number of calls necessary to find the winner?

   Solution: n calls are necessary to find the winner if the n-th guess is correct and there are exactly two correct guesses among the first $n-1$. The probability of this is

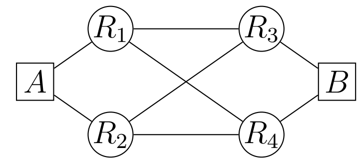
   $$\mathrm{P}(N = n) = \binom{n-1}{2} \cdot 0.7^3 \cdot 0.3^{n-3}.$$

(b) What is the probability that the station will need five or more calls to find a winner?

Solution: No winner has been found in the first four calls if the number of correct guesses in those calls is 0, 1, or 2. The probability of this is

$$P(N \geq 5) = 0.3^4 + 4 \cdot 0.7 \cdot 0.3^3 + \binom{4}{2} \cdot 0.7^2 \cdot 0.3^2 = 0.3483.$$

3. Computers A and B are linked through routers R_1 to R_4 as in the picture. Each router fails independently with probability 10%.



(a) What is the probability there is a connection between A and B?

Solution: Let R_i be the event that router i is operational. The event "there is a connection between A and B" is $(R_1 \cup R_2) \cap (R_3 \cup R_4)$. By independence

$$\begin{aligned} P((R_1 \cup R_2) \cap (R_3 \cup R_4)) &= P(R_1 \cup R_2)\, P(R_3 \cup R_4) \\ &= (1 - P(R_1^c \cap R_2^c))(1 - P(R_1^c \cap R_2^c)) \\ &= (1 - P(R_1^c)\, P(R_2^c))(1 - P(R_3^c)\, P(R_4^c)) \\ &= (1 - 0.1^2)^2 \\ &= 0.9801. \end{aligned}$$

(b) Are the events "there is a connection between A and B" and "exactly two routers fail" independent? Justify your answer.

Solution: No. The probability that there is a connection between A And B given that exactly two routers fail is 2/3: Given that exactly two routers fail, the failed routers are equally likely to be any of the 6 pairs $R_1R_3, R_1R_4, R_2R_3, R_2R_4, R_1R_2, R_3R_4$, and there is a connection between A and B in the first 4 out of these 6 possibilities. This probability is not equal to the unconditional probability from part (a) and so the two events are not independent.

4. A bus takes you from A to B in 10 minutes. On average a bus comes once every 5 minutes. A taxi takes you in 5 minutes, and on average a taxi comes once every 10 minutes. Their arrival times are independent exponential random variables. A bus comes first.

(a) If you want to minimize the (expected) travel time, should you take this bus?

Solution: Yes. If you waited for a taxi your expected travel time would be the expected waiting time for the next taxi which is 10 minutes plus its travel time which is another 5 minutes for a total of 15 minutes.

(b) If you do take the bus, what is the probability that you made the wrong decision?

Solution: The probability of a wrong decision is the probability that a taxi arrives within the next five minutes, which is the probability that an Exponential(1/10) random variable is less than 5, which is $1 - e^{-5/10} = 1 - e^{-1/2} \approx 39.35\%$.

5. 10 people toss their hats and each person randomly picks one. The experiment is repeated one more time.

(a) What is the probability that Bob picked his own hat both times?

Solution: By independence, the probability that Bob picked his hat both times is the product of the probabilities that he picked it in each trial, so it is $(1/10) \cdot (1/10) = 1/100$.

(b) Let A be the event that at least one person picked their own hat both times. True or false: $P(A) > 25\%$? Justify your answer.

Solution: False. Let X_i take value 1 if person i picked their hat both times. A occurs if $X = X_1 + \cdots + X_{10} \geq 1$. By part (a) and linearity of expectation, $\mathrm{E}[X] = 10 \cdot (1/100) = 0.1$. By Markov's inequality, $\mathrm{P}[X \geq 1] \leq \mathrm{E}[X]/1 = 0.1$ which is less than 25%.

6. You are offered to play the following game for \$2: Toss three 3-sided dice and collect D dollars, where D is the number of distinct face values that appear.

(a) Is it rational for you to play this game? Justify your answer.

Solution: Yes. We first calculate the PMF of D. Let X, Y, and Z be the face values of the three dice. By the multiplication rule,

$$\mathrm{P}(D = 1) = \mathrm{P}(X = Y = Z) = \mathrm{P}(Y = X)\,\mathrm{P}(Z = X \mid Y = X) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$
$$\mathrm{P}(D = 3) = \mathrm{P}(X, Y, Z \text{ distinct}) = \mathrm{P}(Y \neq X)\,\mathrm{P}(Z \neq Y, X \mid Y \neq X) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$

so $\mathrm{P}(D = 2) = 1 - 1/9 - 2/9 = 1/3$. The profit from the game is $P = D - 2$ and $\mathrm{E}[P] = 2/9 - 1/9 = 1/9$, which is positive so playing the game has positive utility.

(b) Apply the Central Limit Theorem to estimate the probability that you come out ahead (you earn some money) after playing 90 rounds of the game.

Solution: The variance $\mathrm{Var}[P]$ of the profit is

$$\mathrm{Var}[P] = \mathrm{E}[P^2] - \mathrm{E}[P]^2 = \frac{1}{9} \cdot (-1)^2 + \frac{2}{9} \cdot 1^2 - \left(\frac{1}{9}\right)^2 = \frac{1}{3} - \frac{1}{81} = \frac{26}{81}.$$

Let $X = P_1 + \cdots + P_{90}$ be the profit after 90 rounds. As $P_1, \ldots, P_{90}$ are independent and have the same PMF, the CDF of X can be approximated by the CDF of a normal random variable with mean $\mathrm{E}[X] = 90 \cdot 1/9 = 10$ and variance $\mathrm{Var}[X] = 90 \cdot 26/81 = 260/9$. Therefore

$$\mathrm{P}(X > 0) \approx \mathrm{P}(10 + Z \cdot \sqrt{260/9} > 1/2) = \mathrm{P}(Z > -9.5/\sqrt{260/9}) \approx \mathrm{P}(Z > -1.767) \approx 0.960$$

for a $\mathrm{Normal}(0, 1)$ random variable Z. (The $1/2$ comes from the continuity criterion. Plugging in zero instead of $1/2$ gives a similar answer.)