Practice Midterm 1

1. 3 red balls and 3 blue balls are randomly arranged on a line. Let X be the position of the first blue ball. (E.g. for the arrangement `RBRBBR`, $X = 2$.) Find the probability mass function of X.

   Solution: The sample space consists of all arrangements of 3 red balls and 3 blue balls. We assume equally likely outcomes. The random variable X takes integer values from 1 to 4. X takes value x when the first $x - 1$ balls are red and the x-th ball is blue; the remaining $6 - x$ balls must then contain exactly two blue balls. Using the equally likely formula, we get that

$$P(X = x) = \frac{\binom{6-x}{2}}{\binom{6}{3}} = \frac{(6-x)(5-x)}{40}$$

   or, in tabular form,

x	1	2	3	4
$P(X = x)$	1/2	3/10	3/20	1/20

2. Half the students know the answer to a true-false question. The other half guesses at random. I ask a random student and his answer is correct. What is the probability he knows the answer?

   Solution: The sample space consists of all students under equally likely outcomes. Let K be the event a student knows the answer, C be the event his answer is correct. We have $\mathrm{P}(K) = 1/2, \mathrm{P}(C \mid K) = 1, \mathrm{P}(C \mid K^{\mathsf{c}}) = 1/2$. By Bayes' rule

$$\mathrm{P}(K \mid C) = \frac{\mathrm{P}(C \mid K)\,\mathrm{P}(K)}{\mathrm{P}(C \mid K)\,\mathrm{P}(K) + \mathrm{P}(C \mid K^{\mathsf{c}})\,\mathrm{P}(K^{\mathsf{c}})}.$$

   Plugging in the values we get $\mathrm{P}(K \mid C) = 2/3$.

3. Each pair of computers a, b and c is linked via a cable that fails with probability 10%. Their failures are independent. Let C_{xy} be the event C_{xy} is the event "there is a working connection between computers x and y." Are C_{ab} and C_{bc} independent?

   Solution: No. Let F_{xy} be the event "cable xy fails." It is easier to work with the complement events C_{xy}^c meaning "there is no working connection between computers x and y." By independence

$$\begin{aligned}
\mathrm{P}(C_{ab}^c) &= \mathrm{P}(F_{ab} \cap (F_{ac} \cup F_{bc})\\
&= \mathrm{P}(F_{ab})\,\mathrm{P}(F_{ac} \cup F_{bc})\\
&= \mathrm{P}(F_{ab})(1 - \mathrm{P}(F_{ac}^c)\,\mathrm{P}(F_{bc}^c))\\
&= 0.1 \cdot (1 - 0.9^2)\\
&= 0.019.
\end{aligned}$$

By symmetry $\mathrm{P}(C_{bc}^c)$ is also 0.019. The event $C_{ab}^c \cap C_{bc}^c$ happens exactly when both cables ab and bc fail, so $\mathrm{P}(C_{ab}^c \cap C_{bc}^c) = \mathrm{P}(F_{ab} \cap F_{bc}) = \mathrm{P}(F_{ab})\,\mathrm{P}(F_{bc}) = 0.1^2 \neq 0.019^2 = \mathrm{P}(C_{ab}^c)\,\mathrm{P}(C_{bc}^c)$.

It is possible to reach this conclusion without any calculation. The event F_{ab} contains the event C_{ab}^c: If there is no connection between a and b, cable ab must fail. Moreover, the intersection $F_{ab} \cap C_{ab}$ has nonzero probability: It is possible that cable ab fails and there is still a connection. By the axioms of probability, $\mathrm{P}(F_{ab}) > \mathrm{P}(C_{ab}^c)$. For the same reason $\mathrm{P}(F_{bc}) > \mathrm{P}(C_{bc}^c)$. However, $C_{ab}^c \cap C_{bc}^c$ (no connection between ab and no connection between bc) is equal to $F_{ab} \cap F_{bc}$ so

$$\mathrm{P}(C_{ab}^c \cap C_{bc}^c) = \mathrm{P}(F_{ab} \cap F_{bc}) = \mathrm{P}(F_{ab})\,\mathrm{P}(F_{bc}) > \mathrm{P}(C_{ab}^c)\,\mathrm{P}(C_{bc}^c).$$

4. The average lifetime of a lightbulb is 10 months. You install 10 lightbulbs on January 1. What is the probability that at least one of them fails in January? Assume their failures are independent.

   Solution: There are two acceptable answers to this question. We can model the month in which a given lightbulb fails as a Geometric(p) random variable. Its expectation is 10, so $p = 1/10$. The probability that a given lightbulb doesn't fail in January is then $1 - p = 9/10$. The probability that none of the 10 fails this month is $(9/10)^{10}$. The probability that at least one of them fails this month is $1 - (9/10)^{10} \approx 0.651$.

   We can also model the number of times we have to replace a given lightbulb within a month as a Poisson(λ) random variable. The rate is one per 10 months, so $\lambda = 1/10$ per month. The probability that this lightbulb doesn't fail in January is then $\mathrm{P}(\text{Poisson}(1/10) = 0) = e^{-1/10}$. The probability that none of the 10 fails in January is $(e^{-1/10})^{10} = 1/e$, so the probability that at least one fails in January is $1 - 1/e \approx 0.632$.

5. Eight people's hats are mixed up and randomly redistributed. What is the expected number of pairs that exchanged hats (Alice got Bob's and Bob got Alice's)?

   Solution: For every two people i and j we introduce a random variable X_{ij} that takes value 1 if the two exchanged hats and 0 if not. The expected value of X_{ij} is the probability that i and j exchanged hats, which is $1/8 \cdot 7$: Bob gets Alice's hat with probability 1/8, and given this happened Alice gets Bob's with probability 1/7. The number of pairs that exchanged hats is $X_{12} + X_{13} + \cdots + X_{78}$, where the indices range over all (ordered) $\binom{8}{2}$ pairs of people. By linearity of expectation,

$$\mathrm{E}[X_{12} + X_{13} + \cdots + X_{78}] = \mathrm{E}[X_{12}] + \mathrm{E}[X_{13}] + \cdots + \mathrm{E}[X_{78}] = \binom{8}{2} \cdot \frac{1}{8 \cdot 7} = \frac{1}{2}.$$

Practice Midterm 2

1. Alice, Bob, Charlie, and Dave are randomly seated at a round table. The probability that Alice is seated next to Bob is 70%. The probability that Bob is seated next to Charlie is 40%. What is the probability that Charlie is seated next to Alice?

   Solution: Let A be the event "Alice sits across from Dave". Then the complement of A is the event "Bob sits next to Charlie", so $\mathrm{P}(A) = 1 - \mathrm{P}(A^c) = 1 - 0.4 = 0.6$. Define B and C analogously with Bob and Charlie replacing Alice, respectively. By the same reasoning $\mathrm{P}(C) = 1 - 0.7 = 0.3$. Since A, B, and C partition the sample space, $\mathrm{P}(B) = 1 - \mathrm{P}(A) - \mathrm{P}(C) = 0.1$. Therefore the event B^c representing "Charlie sits next to Alice" has probability $1 - \mathrm{P}(B) = 0.9$.

2. An unknown number of independent trials is performed, each of which succeeds with the same probability. You can only observe the number of successful trials. After many runs of this experiment you conclude that the expected number of successful trials is 6, and the variance of this number is 2. How many trials are performed?

   Solution: Let X denote the number of successful trials. Since each trial is performed independently and succeeds with the same probability. X is Binomial(n, p). We have $\mathrm{E}[X] = np = 6$ and $\mathrm{Var}[X] = np(1-p) = 2$. Solving the two equations gives us $p = \frac{2}{3}$ and $n = 9$.

3. Alice has a \$1, a \$2, and a \$5 coin. She randomly and secretly picks a coin with each hand (with equal probabilities) and shows the coin in her left hand to Bob. Bob may keep this coin or switch to the coin in Alice's right hand. Assuming Bob plays optimally what is his expected utility?

   Solution: Let L and be the value of the coin in Alice's left and hands, respectively. The expectation of R given L is the average value of the other two coins:

$$\begin{aligned}
\mathrm{E}[R|L=1] &= \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 5 = 3.5 \\
\mathrm{E}[R|L=2] &= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 5 = 3 \\
\mathrm{E}[R|L=5] &= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 2 = 1.5
\end{aligned}$$

   To maximize his expected utility, Bob should keep the coin when $L = 5$ and switch when $L = 1$ or $L = 2$. Since all values of L are equally likely, Bob's expected utility is

$$\frac{1}{3}\cdot 3.5 + \frac{1}{3}\cdot 3 + \frac{1}{3}\cdot 5 = \frac{23}{6} \approx 3.833.$$

4. Trains reach Kowloon Station at an average rate of two trains per hour. Bob observed at least one train reach the station within the last hour. Given this information, what is the expected number of trains that reached the station within the last hour?

   Solution: If we model the train arrivals in the last hour as a Poisson(2) random variable N, we are looking for the expectation of N given $N > 0$. By the total expectation formula

$$\mathrm{E}[N] = \mathrm{E}[N|N>0]\,\mathrm{P}(N>0) + \mathrm{E}[N|N=0]\,\mathrm{P}(N=0).$$

   We know that $\mathrm{E}[N] = 2$, $\mathrm{E}[N|N=0] = 0$, and $\mathrm{P}(N=0) = e^{-2}\cdot 2^0/0! = e^{-2}$. By the axioms $\mathrm{P}(N>0) = 1 - \mathrm{P}(N=0) = 1 - e^{-2}$, so

$$\mathrm{E}[N|N>0] = \frac{\mathrm{E}[N]}{\mathrm{P}(N>0)} = \frac{2}{1-e^{-2}} \approx 2.313.$$

5. A dealer divides ten cards with face values $1, 2, \ldots, 10$ among five players. Each player is randomly assigned two cards. A player wins if the sum of his cards' face values is 17 or higher. What is the expected number of winners? (**Extra credit:** What is the probability that there are no winners?)

   Solution: The number W of winners is $W_1 + W_2 + W_3 + W_4 + W_5$ where W_i is an indicator random variable for the event "player i wins". We calculate $\mathrm{E}[W_i] = \mathrm{P}(W_i = 1)$ using the equally likely outcomes formula by which W_i is the number of pairs of cards whose sum is 17 or higher divided by the total number of pairs. The total number of pairs is $\binom{10}{2} = 45$, while the pairs of value 17 or higher are $\{7, 10\}, \{8, 9\}, \{8, 10\}, \{9, 10\}$. Therefore $\mathrm{P}(W_i = 1) = 4/45$ and $\mathrm{E}[W] = 5 \cdot 4/45 = 4/9$.

For the extra credit part, it is difficult to calculate $\mathrm{P}(W = 0)$ directly. It is easier to get a handle on the event $W = 2$: It the intersection of A and B, where A is "some player's cards are 7 and 10" and B is "some player's cards are 8 and 9". As $\mathrm{P}(A) = 1/9$ and $\mathrm{P}(B|A) = 1/7$, $\mathrm{P}(W = 2) = 1/63$. Since $\mathrm{E}[W] = \mathrm{P}(W = 1) + 2\,\mathrm{P}(W = 2)$ we get that $\mathrm{P}(W = 1) = 4/9 - 2 \cdot 1/63 = 26/63$. As the PMF of W must add up to one, $\mathrm{P}(W = 0) = 1 - 1/63 - 26/63 = 4/7$.