There are 12 chopsticks of which 4 are black and 8 are white. Each of 6 diners is assigned a random pair of chopsticks. What is the expected number of diners whose chopsticks are of different color?

Solution: The number X of diners with chopsticks of different colors is a sum of indicator random variables $X_1 + X_2 + \cdots + X_6$, where X_i is the indicator for the event that the i-th diner has a black and a white chopstick. Conditioning on the color of their left chopstick, using the total probability theorem we find that $\mathrm{E}[X_i] = \mathrm{P}(X_i = 1) = 4/12 \cdot 8/11 + 8/12 \cdot 4/11 = 16/33$. By linearity of expectation $\mathrm{E}[X] = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + = 6 \cdot 16/33 = 32/11 \approx 2.91$.

Alternative solution: X can take only values 0, 2, or 4. We calculate its PMF. It will be helpful to think of the black chopsticks as being assigned one by one to a random hand of a random diner.

The event $X = 4$ happens when the black chopsticks are all assigned to different diners. Let B_i be the event that the first i black chopsticks are assigned to different diners. Then $\mathrm{P}(B_2) = 10/11$ because after the first black chopstick has been assigned, the second one is equally likely to land in the remaining 11 hands, out of which 10 belong to other diners. For a similar reason $\mathrm{P}(B_3|B_2) = 8/10$: Given that the first two were assigned to different diners, there are 10 hands left to pick up the third one out of which 8 belong to other diners. Continuing we get $\mathrm{P}(B_4|B_3) = 6/9$. By the multiplication rule $\mathrm{P}(X = 4) = \mathrm{P}(B_4) = \mathrm{P}(B_2)\,\mathrm{P}(B_3|B_2)\,\mathrm{P}(B_4|B_3) = 10/11 \cdot 8/10 \cdot 6/9 = 16/33$.

The event $X = 0$ happens when two diners pick up all the black chopsticks. Suppose the first two black chopsticks were already assigned. Conditioned on B_2, these two went to different diners, so the remaining two must be assigned to the other two hands of these diners. This happens in 2 out of 10 ways for the third black chopstick and 1 out of 9 for the fourth one, so $\mathrm{P}(X = 0|B_2) = 2/10 \cdot 1/9$. Conditioned on B_2^c, the fourth chopstick must be assigned to the same person as the third one, which can happen in 1 out of 9 ways. Therefore $\mathrm{P}(X = 0|B_2^c) = 1/9$. By the total probability formula,

$$\mathrm{P}(X = 0) = \mathrm{P}(X = 0|B_2)\,\mathrm{P}(B_2) + \mathrm{P}(X = 0|B_2^c)\,\mathrm{P}(B_2^c) = 2/10 \cdot 1/9 \cdot 10/11 + 1/9 \cdot 1/11 = 1/33.$$

(An alternative way to calculate $\mathrm{P}(X = 0)$ is to write $X = 0$ as a disjoint union of three events $A_1 \cup A_2 \cup A_3$, where A_i consists of those assignments where the ith and 4th black chopstick are assigned to one diner and the other two are assigned to another diner. By the multiplication rule $\mathrm{P}(A_i) = 1/11 \cdot 1/9$, so $\mathrm{P}(X = 0) = 1/33$.)

By the axioms $\mathrm{P}(X = 2) = 1 - 16/33 - 1/33 = 16/33$. Therefore $\mathrm{E}[X] = 2 \cdot 16/33 + 4 \cdot 16/33 = 32/11$.