1. Alice, Bob, and Charlie hold a lucky draw for two tickets to a concert with the following odds:

   - The probability that Alice gets one of the tickets is 60%.
   - The probability that Bob gets one of the tickets is 70%.

   What is the probability that Alice and Bob both get tickets?

   Solution: The sample space consists of the three outcomes $\{ab, ac, bc\}$, where ab represents Alice and Bob getting the tickets, and so on. Denote their probabilities by p_{ab}, p_{ac}, and p_{bc}. The event "Alice gets one of the tickets" is $\{ab, ac\}$ so $p_{ab}+p_{ac} = 0.6$. Similarly $p_{ab}+p_{bc} = 0.7$. Since the probabilities must add up to one,

   $$p_{ab} = (p_{ab} + p_{ac}) + (p_{ab} + p_{bc}) - 1 = 0.6 + 0.7 - 1 = 0.3.$$

   Alternative solution: Let A, B, and C be the events "Alice gets a ticket" and so on. By the axioms the complementary events have probabilities $\mathrm{P}(A^c) = 0.4$ and $\mathrm{P}(B^c) = 0.3$. The events A^c, B^c, and C^c partition the sample space (they are disjoint and exactly one of the three must occur) so their probabilities must add up to one. Therefore $\mathrm{P}(C^c) = 1-0.4-0.3 = 0.3$. The event C^c happens exactly when Alice and Bob both get tickets, so the desired probability is 30%.

2. Alice flips seven fair coins. Let H be the event that the last flip is a head and A be the event that at least one flip is a head. Calculate (a) $\mathrm{P}(A^c)$; (b) $\mathrm{P}(H|A)$; (c) $\mathrm{P}(A|H)$.

   Solution: The sample space consists of all sequences of seven heads or tails, namely $\Omega = \{\mathtt{H}, \mathtt{T}\}^7$. All outcomes are equally likely.

   (a) A^c is the event "all flips are tails", so $\mathrm{P}(A^c) = |A^c|/2^7 = 1/128$.

   (b) From the complement rule $\mathrm{P}(A) = 1 - \mathrm{P}(A^c) = 127/128$. Then

   $$\mathrm{P}(H|A) = \frac{\mathrm{P}(H \cap A)}{\mathrm{P}(A)} = \frac{\mathrm{P}(H)}{\mathrm{P}(A)} = \frac{64}{127}.$$

   (c) Now $\mathrm{P}(A|H) = \mathrm{P}(A \cap H)/\mathrm{P}(H) = \mathrm{P}(H)/\mathrm{P}(H) = 1$. This is sensible: Given that the last flip is a head, at least one must be a head with probability 1.

3. The Los Angeles Lakers and the Boston Celtics play one game in each city. Each team wins their home game with 70% probability. There is a 40% probability that both win their home games. The Lakers win their home game. What is the probability that they win in Boston?

   Solution: Let L and C be the events that the Lakers and Celtics win their home games, respectively. We are told that $\mathrm{P}(L) = \mathrm{P}(C) = 0.7$ and $\mathrm{P}(L \cap C) = 0.4$. We are interested in $\mathrm{P}(C^c|L) = \mathrm{P}(L \cap C^c)/\mathrm{P}(L)$. By the axioms of probability $\mathrm{P}(L \cap C^c) = \mathrm{P}(L) - \mathrm{P}(L \cap C) = 0.7 - 0.4 = 0.3$), so $\mathrm{P}(C^c|L) = 0.3/0.7 = 3/7 \approx 0.429$.

4. There are 6 red balls and 1 blue ball. Each ball is randomly placed in one of two bins.

   (a) What is the probability that the bin with the larger number of balls contains k balls ($k \in \{4, 5, 6, 7\}$)?

(b) What is the probability that the bin with the larger number of balls contains 3 red balls and 1 blue ball?

Solution:

(a) The sample space Ω consists of all sequences of length 7, where the value of each position can be either `1` or `2`, denoting which bin the ball goes to. Ω has size 2^7. Let E_k denote the event the bin with the larger number of balls contains k balls. Then E_k consists of strings that contain k `1`s and $7-k$ `2`s, or k `2`s and $7-k$ `1`s. Therefore $|E_k| = (\binom{7}{k} + \binom{7}{7-k})/2^7 = \binom{7}{k}/2^6$. By the equally likely outcomes formula,

k	4	5	6	7
$P(E_k)$	$\frac{35}{64}$	$\frac{21}{64}$	$\frac{7}{64}$	$\frac{1}{64}$

(b) Let A be the event of interest. We represent the blue ball by the first position in the sequence. Then A consists of those sequences that start with a `1` and have exactly four `1`s or start with a `2` and have exactly four `2`s. By the generalized multiplication rule, $|A| = 2 \cdot \binom{6}{3} = 40$. By the equally likely outcomes formula $\mathrm{P}(A) = 40/2^7 = 5/16$.

An alternative solution is to use the multiplication rule for conditional probabilities: $\mathrm{P}(A) = \mathrm{P}(A|E_4)\,\mathrm{P}(E_4)$. Conditioning the outcome on E_4 means drawing exactly four balls in the larger bin. Therefore $\mathrm{P}(A|E_4)$ is the probability of drawing a blue ball given that exactly four balls were drawn (and deposited in the larger bin). This is the fraction of arrangements of the balls in which the blue ball takes one of the first four positions so $\mathrm{P}(A|E_4) = 4/7$ and $\mathrm{P}(A) = \frac{4}{7} \cdot \frac{35}{64} = 5/16$.