1. Cup 1 has 3 blue balls and 1 red ball. Cup 2 has 1 blue ball and 3 red balls. You choose a cup at random and draw three balls from it without replacement. Let C_i and B_j be the events "you chose cup i" and "the j-th ball is blue", respectively. Is it true that:

   (a) B_1 and B_2 are independent given C_1.

   (b) B_1 and B_2 are independent.

   (c) B_2 and B_3 are independent given B_1.

   Solution:

   (a) **False.** We expect the color of the first ball to affect the color of the second ball, namely $\mathrm{P}(B_2|C_1) \neq \mathrm{P}(B_2|B_1 \cap C_1)$. By symmetry $\mathrm{P}(B_2|C_1) = \mathrm{P}(B_1|C_1) = 3/4$. However $\mathrm{P}(B_2|B_1 \cap C_1) = 2/3$ as there will be 2 blue balls and 1 red ball left after a blue ball is taken out.

   (b) **True.** By the total probability theorem

   $$\mathrm{P}(B_1) = \mathrm{P}(B_1 \mid C_1)P(C_1) + P(B_1 \mid C_2)P(C_2) = \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{2}$$

   By symmetry, $\mathrm{P}(B_2)$ is also $1/2$. As C_2 doesn't have enough blue balls, $B_1 \cap B_2 = B_1 \cap B_2 \cap C_1$ and by the multiplication rule

   $$\mathrm{P}(B_1 \cap B_2) = \mathrm{P}(B_1 \cap B_2 \mid C_1)\,\mathrm{P}(C_1) = \mathrm{P}(B_2|B_1 \cap C_1)\,\mathrm{P}(B_1|C_1)\,\mathrm{P}(C_1) = \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{4}.$$

   so $\mathrm{P}(B_1 \cap B_2) = \mathrm{P}(B_1) \cdot \mathrm{P}(B_2)$.

   (c) **True.** By part (b) (and symmetry of the blue balls) $\mathrm{P}(B_3|B_1) = \mathrm{P}(B_3) = 1/2$. On the other hand, $B_1 \cap B_2$ means C_1 must hold so

   $$\mathrm{P}(B_3|B_1 \cap B_2) = \mathrm{P}(B_3|B_1 \cap B_2 \cap C_1) = \frac{1}{2}$$

   because after drawing two blue balls out of cup one there is an even chance that the third ball is also blue.

   Alternative solution: The events B_1, B_2, B_3 are independent. From part (b) we know that any pair of them is independent. As for all three, $\mathrm{P}(B_1 \cap B_2 \cap B_3) = \mathrm{P}(B_1 \cap B_2 \cap B_3|C_1)\,\mathrm{P}(C_1) = 1/4 \cdot 1/2 = 1/8$ which is the same as $\mathrm{P}(B_1)\,\mathrm{P}(B_2)\,\mathrm{P}(B_3)$. Therefore $\mathrm{P}(B_2 \cap B_3|B_1) = \mathrm{P}(B_2 \cap B_3) = \mathrm{P}(B_2)\,\mathrm{P}(B_3) = \mathrm{P}(B_2|B_1)\,\mathrm{P}(B_3|B_1)$.

2. Suppose the number of school bus arriving at the Sir Run Run Shaw Hall in any time interval is a Poisson random variable, with a rate of 1 bus in 5 minutes.

   (a) What is the probability that no bus arrives in the next 30 minutes?

   Solution: The rate of bus arrivals is 6 in 30 minutes, so the number of buses that arrive in a 30-minute interval is a Poisson(6) random variable X. We are interested in the probability of the event $X = 0$, which equals $e^{-6} \approx 0.0024$.

(b) What is the probability that at least 5 buses arrive in the next 10 minutes?

Solution: The rate of arrivals is 2 in 10 minutes, so we want to know what is the probability that a Poisson(2) random variable Y takes value 5 or more. So we need to calculate

$$P(Y \geq 5) = P(Y = 6) + P(Y = 7) + \cdots,$$

which is an infinite sum. By the axioms of probability, we can instead calculate

$$\begin{aligned} P(Y \geq 5) &= 1 - P(Y < 5) \\ &= 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) - P(Y = 3) - P(Y = 4) \\ &= 1 - \frac{e^{-2} \cdot 2^0}{0!} - \frac{e^{-2} \cdot 2^1}{1!} - \frac{e^{-2} \cdot 2^2}{2!} - \frac{e^{-2} \cdot 2^3}{3!} - \frac{e^{-2} \cdot 2^4}{4!}, \\ &= 1 - 7e^{-2} \end{aligned}$$

which is about 0.0527.

(c) Let M be the minute in which the first bus arrives. For example if the first bus arrives in 4 min 23 sec then $M = 5$. What kind of random variable is M? What is $P(M = 6)$?

Solution: Let E_t be the event "at least one bus arrives in the t-th minute". Then $E_1, E_2, E_3, \ldots$ are independent. The number of buses that arrive in a given minute is a Poisson(1/5) random variable, so $P(E_i) = 1 - P(E_i^c) = 1 - e^{-1/5} \approx 0.181$. Then M is the first minute when E_M happens so it is a Geometric$(1 - e^{-1/5})$ random variable. Therefore

$$P(M = 6) = (e^{-1/5})^5(1 - e^{-1/5}) = 1/e - 1/e^{5/6} \approx 0.067.$$

3. You attend a wedding with 500 guests.

(a) Let X be the number of other guests that share your birthday. What kind of random variable is X?

(b) What is the probability that $X = 1$?

(c) Now model the number of other guests that share your birthday as a Poisson(λ) random variable N. What is the rate λ? How do $P(X = 1)$ and $P(N = 1)$ compare?

Solution:

(a) We can model the number of guests having *your* birthday as a Binomial($n = 499, p = 1/365$) random variable X.

(b) The probability that $X = 1$ is $\binom{499}{1} \cdot p \cdot (1 - p)^{499-1} \approx 0.3487$.

(c) We can model this process as a Poisson(λ) random variable N with $\lambda = np = 499/365$. Then the probability of $N = 1$ is $\lambda \cdot e^{-\lambda} \approx 0.3484$, which is slightly smaller than the probability of $X = 1$.

4. Calculate the PMFs of the following random variables:

(a) The first time X at which both the patterns `TH` and `HT` have appeared in a sequence of fair coin flips. For example, $X = 6$ for the sequence `HHTTTH`.

Solution: The events $X = 1$ and $X = 2$ never occur. For $x \geq 3$, the event $X = x$ occurs when either there is a head at time x, and the first $x - 1$ flips consist of some heads followed by some tails with at least one of each kind, or vice versa. As there are $x - 2$ possible positions in which the first flip can occur, there are $x - 2$ possible outcomes of each kind. As each outcome has probability 2^{-x}, $P(X = x) = 2(x - 2)/2^x$ for $x \geq 3$.

(b) The first time Y at which all three face values have appeared in a sequence of rolls of a fair 3-sided die. For example, $Y = 6$ for the sequence `232231`.

Solution: The events $Y = 1$ and $Y = 2$ never happen. For $y \geq 3$, let R_y be the value of the y-th roll. Conditioned on $R_y = 1$, the event $Y = y$ happens when the first $y - 1$ rolls are all `2`s or `3`s with at least one of each type. Let A be the event "the first $y - 1$ rolls are all `2`s or `3`s" and B be the event "the first y rolls are all of the same type". Then $\mathrm{P}(A) = (2/3)^{y-1}$ by independence of the rolls and $\mathrm{P}(B) = 2/3^{y-1}$ as there are only two outcomes in B. By the axioms of probability,

$$\mathrm{P}(Y = y|R_y = 1) = \mathrm{P}(A \cap B^c) = \mathrm{P}(A) - \mathrm{P}(A \cap B) = (2/3)^{y-1} - 2/3^{y-1}.$$

As $Y = y$ and $R_y = 1$ are independent events, $\mathrm{P}(Y = y) = (2/3)^{y-1} - 2/3^{y-1}$ for $y \geq 3$.