1. A point is chosen uniformly at random inside a circle with radius 1. Let X be the distance from the point to the centre of the circle. What is the (a) CDF (b) PDF (c) expected value and (d) variance of X? *[Adapted from textbook problem 3.2.7]*

   Solution:

   (a) The PDF of the point is uniform over the circle which has area π, so it has value $1/\pi$ inside the center and zero outside. The event $X \leq x$ consists of all the points in the circle that are at distance less than or equal to x from the center, which is itself a circle of radius x. Therefore the CDF is $\mathrm{P}(X \leq x) = 1/\pi \times x^2\pi = x^2$, and the PDF is $f_X(x) = d\,\mathrm{P}(X \leq x)/dx = 2x$ for $0 \leq x \leq 1$.

   (b) The expected value of X is $\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} 2x^2 = \frac{2}{3}x^3 \mid_0^1 = \frac{2}{3}$.

   (c) The variance of X is $\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2$, where

   $$\mathrm{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_{-\infty}^{\infty} 2x^3 = \tfrac{1}{2}x^4 \mid_0^1 = \frac{1}{2}.$$

   Therefore, $\mathrm{Var}(X) = 14 - (\frac{2}{3})^2 = \frac{1}{18}$.

2. Bob's arrival time at a meeting with Alice is X hours past noon, where X is a random variable with PDF

   $$f(x) = \begin{cases} cx, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the value of the constant c.

   Solution: By the axioms of probability, $\int_{-\infty}^{\infty} f(x)dx = 1$. Since

   $$\int_{-\infty}^{\infty} f(x) = \tfrac{1}{2}cx^2 \mid_0^1 = \tfrac{1}{2}c,$$

   so c must be equal to 2.

   (b) What is the probability that Bob arrives by 12.30?

   Solution: The CDF is $F(x) = \mathrm{P}(X \leq x) = \int_{-\infty}^{\infty} f(x)dx = x^2$, so $\mathrm{P}(X \leq 0.5) = 0.25$.

   (c) What is the expected time of Bob's arrival?

   Solution: The expected value is $\mathrm{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} 2x^2dx = \frac{2}{3}x^3 \mid_0^1 = \frac{2}{3}$. So Bob is expected to arrive at 12.40.

   (d) Given that Bob hasn't arrived by 12.30, what is the probability that he arrives by 12.45?

   Solution: The probability that Bob hasn't arrived by 12:30 is $P(X > 0.5) = 1 - P(X \leq 0.5) = 1 - 0.25 = 0.75$. The probability that Bob hasn't arrived by 12:30 but arrives by 12:45 is $P(0.75 \geq X > 0.5) = P(X \leq 0.75) - P(X \leq 0.5) = F(0.75) - F(0.5) = \frac{9}{16} - \frac{1}{4} = \frac{5}{16}$. Therefore, the conditional probability is

   $$\mathrm{P}(X \leq 0.75 \mid X > 0.5) = \frac{\mathrm{P}(0.75 \geq X > 0.5)}{\mathrm{P}(X > 0.5)} = \frac{5/16}{1 - 1/4} = \frac{5}{12}.$$

(e) Given that Bob hasn't arrived by 12.30, what is the expected hour of Bob's arrival?

Solution: The conditional PDF is

$$f_{X|X>0.5}(x) = \frac{f_X(x)}{\mathrm{P}(X > 0.5)} = \frac{f_X(x)}{0.75} = \begin{cases} \frac{8}{3}x & 0.5 < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The conditional expectation is

$$\mathrm{E}[X|X > 0.5] = \int_{-\infty}^{\infty} x f_{X|X>0.5}(x) = \int_{0.5}^{1} \frac{8}{3}x^2 = \frac{8}{9}x^3|_{0.5}^{1} = \frac{7}{9}.$$

Bob's conditional expected arrival time is about 12.47.

3. The joint PDF of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} C(x + y + 1)y, & \text{if } 0 \leq x \leq 2, 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find (a) the value of C and (b) The conditional PDF $f_{Y|X}(y|x)$.

Solution:

(a) The PDF must integrate to one, so

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x, y)dxdy = \int_0^2 \int_0^2 C(x + y + 1)ydxdy = \frac{40}{3}C.$$

Therefore $C = \frac{3}{40}$.

(b) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y)dy = \int_0^2 C(x + y + 1)ydy = C(2x + \frac{14}{3})$. Using the convolution formula,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{(x + y + 1)y}{2x + \frac{14}{3}}.$$

4. Alice and Bob agree to meet. Alice's arrival time A is uniform between 12:00 and 12:45 and Bob's arrival time B is uniform between 12:15 and 13:00. Let E be the event "Alice and Bob arrive within 30 minutes of one another".

(a) What is $\mathrm{P}(E)$ assuming A and B are independent?

Solution: We can model A as a Uniform$(0, 3/4)$ random variable and B as an independent Uniform$(1/4, 1)$ random variable. Their marginal PMFs are:

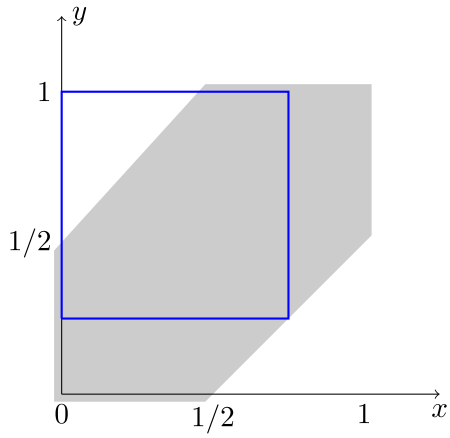
$$f_A(x) = \begin{cases} 4/3 & \text{if } 0 \leq x < 3/4 \\ 0 & \text{otherwise} \end{cases}$$

$$f_B(y) = \begin{cases} 4/3 & \text{if } 1/4 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Because A and B are independent, their joint PMF is

$$f(x, y) = f_A(x)f_B(y) = \begin{cases} 16/9 & \text{if } 0 \leq x < 3/4 \text{ and } 1/4 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The event E is $|B - A| < 1/2$. We want to calculate $\mathrm{P}(E) = \int_E f(x, y)dxdy$.



To understand this expression look at the above picture. The blue square consists of the points (x, y) where $f(x, y)$ is nonzero (it equals $(3/4)^2 = 9/16$). The grey strip are the points x, y such that $|x - y| \leq 1/2$. The intersection of the two consists of the blue square minus the triangle T with vertices $(0, 1/2), (1/2, 1/2), (0, 1)$ so

$$\mathrm{P}(E) = \int_E f(x, y) dxdy = 1 - \int_T \frac{16}{9} dxdy = 1 - \frac{16}{9} \cdot \mathrm{area}(T) = 1 - \frac{16}{9} \cdot \frac{1}{8} = \frac{7}{9}.$$

(b) If you don't know the joint PDF of A and B, how large can $\mathrm{P}(E)$ be?

Solution: If $B = A + 1/4$, then the marginal PMFs of A and B are as in part (a), but $\mathrm{P}(E) = \mathrm{P}(|B - A| < 1/2) = \mathrm{P}(1/4 < 1/2) = 1$.

(c) **(Optional)** If you don't know the joint PDF of A and B, how small can $\mathrm{P}(E)$ be?

Solution: Let E_A be the event $1/4 < A < 3/4$ and E_B be the event $1/4 < B < 3/4$. By the axioms of probability,

$$\mathrm{P}(E) \geq \mathrm{P}(E_A \cap E_B) = \mathrm{P}(E_A) + \mathrm{P}(E_B) - \mathrm{P}(E_A \cup E_B) \geq \mathrm{P}(E_A) + \mathrm{P}(E_B) - 1.$$

As $\mathrm{P}(E_A) = \mathrm{P}(E_B) = 2/3$, $\mathrm{P}(E)$ cannot be smaller than $1/3$.
To argue that $\mathrm{P}(E)$ can be as small as $1/3$ we describe a probability model in which $\mathrm{P}(E) \leq 1/3$. One way to do this is by specifying the marginal PDF of B given A as follows: $B = A + 1/2$ when $0 < A < 1/2$ and $B = 1 - A$ otherwise. It is easy to check that when A is Uniform$(0, 3/4)$ then B is Uniform$(1/4, 1)$, yet $\mathrm{P}(|B - A| < 1/2) = \mathrm{P}(A \geq 1/2) = 1/3$.

5. **(Optional)** Here is a way to solve Buffon's needle problem without calculus. Recall that an ℓ inch needle is dropped at random onto a lined sheet, where the lines are one inch apart.

(a) Let A be the number of lines that the needle hits. Let B be the number of times that a polygon of perimeter ℓ hits a line. Show that $\mathrm{E}[A] = \mathrm{E}[B]$. (**Hint:** Use linearity of expectation.)

Solution: Suppose the polygon has n edges of length $a_1, a_2, \ldots, a_n$. Break up the needle into segments of lengths $a_1, a_2, \ldots, a_n$. Let A_i and B_i be the number of lines hit by the i-th segment of the needle and the i-th edge of the polygon, respectively. Then

$$A = A_1 + \cdots + A_n \quad \text{and} \quad B = B_1 + \cdots + B_n.$$

By linearity of expectation

$$\mathrm{E}[A] = \mathrm{E}[A_1] + \cdots + \mathrm{E}[A_n] \quad \text{and} \quad \mathrm{E}[B] = \mathrm{E}[B_1] + \cdots + \mathrm{E}[B_n].$$

Since the i-th edge of the polygon and the i-th segment of the needle are identical, $\mathrm{E}[A_i] = \mathrm{E}[B_i]$. It follows that $\mathrm{E}[A] = \mathrm{E}[B]$.

(b) Assume that $\ell < \pi$. Calculate the expected number of times that a circle of perimeter ℓ hits a line.

Solution: Let C be the number of times a circle intersects a line. We calculate the PMF of C. Let d be the line segment representing the diameter of the circle that is perpendicular to the lines on the sheet. Since $\ell < \pi$, the length of d is less than 1. The circle hits a line twice if d crosses a line, once if d touches one of the lines, and zero times if d does not intersect any of the lines. The probability that d crosses a line is exactly the length of d, namely ℓ/π, and the probability that d touches a line is zero. Summarizing, the PMF of C is

c	0	1	2
$\mathrm{P}(C = c)$	$1 - \ell/\pi$	0	ℓ/π

Therefore $\mathrm{E}[C] = 2\ell/\pi$.

(c) Assume that $\ell < 1$. Use part (a) and (b) to derive a formula for the probability that the needle hits a line. (**Hint:** The number of hits is an indicator random variable.)

Solution: If we view the circle as a polygon with infinitely many sides, putting together part (a) and (b) we get that $\mathrm{E}[A] = \mathrm{E}[C] = 2\ell/\pi$. Since $\ell < 1$, the number of times the needle intersects a line is a 0/1 valued random variable, so $\mathrm{E}[A] = \mathrm{P}(A = 1) = \mathrm{P}(\text{the needle hits a line})$. Therefore the probability the needle hits a line is exactly $2\ell/\pi$.