1. Raindrops are falling at the rate of 1 drop per second. Let p be the probability of getting more than 120 raindrops within a minute.

   (a) Use Markov's inequality to argue that $p \le 50\%$.

   (b) Use Chebyshev's inequality to argue that $p \le 2\%$.

   (c) Estimate p using the Central Limit Theorem.

   Solution: The number of rain drops per minute X is a Poisson random variable with rate $\lambda = 60$. The mean and the standard derivation of X are $\mu = 60$ and $\sigma = \sqrt{60}$, respectively.

   (a) By Markov's inequality, we have:

   $$\mathrm{P}(X \ge 120) \le \mu/120 = 60/120 = 0.5.$$

   (b) By Chebyshev's inequality, we have:

   $$\mathrm{P}(X \ge 120) \le \mathrm{P}(|X - 60| \ge 60) = \mathrm{P}(|X - \mu| \ge \sqrt{60}\sigma) \le 1/(\sqrt{60})^2 \approx 0.016.$$

   (c) The time T of the 120th raindrop is a sum of 120 independent Exponential(λ) random variables $T_1 + \cdots + T_{120}$. Its expectation and variance are $\mathrm{E}[T] = 120/\lambda = 2$ and $\mathrm{Var}[T] = 120/\lambda^2 = 1/30$. By the Central Limit Theorem,

   $$p = \mathrm{P}(T \le 1) \approx \mathrm{P}(\mathrm{Normal}(2, 1/30) \le 1) = \mathrm{P}(\mathrm{Normal}(0, 1) \le -\sqrt{30}) \approx 2 \cdot 10^{-8}.$$

2. Alice is mailing letters to solicit donations from CUHK alums. From past experience she knows that 30% of the alums make a 500 dollar donation and 10% of the alums make a $1,000$ dollar donation. She needs to mail enough letters to collect 50,000 dollars. Use the Central Limit Theorem to estimate

   (a) the probability she meets her target by mailing 180 letters.

   (b) the number of letters she has to mail to meet her target with 90% probability.

   Solution: The alumni donations $X_1, \dots, X_n$ are independent random variables with mean $\mu = \mathrm{E}[X_i] = 500 \cdot 0.3 + 1000 \cdot 0.1 = 250$ dollars and variance $\sigma^2 = \mathrm{Var}[X_i] = 500^2 \cdot 0.3 + 1000^2 \cdot 0.1 - 250^2 = 112500$ dollars squared. By the Central Limit Theorem we approximate their sum $X = X_1 + \cdots + X_n$ by a normal random variable with mean μn and variance $\sigma^2 n$.

   (a) $\mathrm{P}(X \ge 50,000) \approx \mathrm{P}(\mathrm{Normal}(180 \cdot 250, 180 \cdot 112,500) \ge 50,000) = \mathrm{P}(\mathrm{Normal}(0, 1) \ge 10/9) \approx 0.1335$.

   (b) A $\mathrm{Normal}(0, 1)$ random variable takes values -1.2816 or higher 90% of the time. To achieve the desired probability of success, we need to pick n so that $\mu n - 1.2816\sigma\sqrt{n}$ is at least $50,000$ dollars. Plugging in the values of μ and σ we obtain the inequality

   $$250n - 1.2816 \cdot \sqrt{112,500} \cdot \sqrt{n} \ge 50,000.$$

   By the quadratic formula, it is enough to choose $n = 226$ for this. Hence, Alice needs to mail about 226 alums.

3. The following exam statistics are posted on the course website:

section	no. students	average grade	std. dev.
A	30	65	5
B	20	70	10

what can you say about the number of students whose exam grade was 30 or below?

Estimate the quantity of your interest using (a) Markov's inequality, (b) Chebyshev's inequality and (c) the Central Limit Theorem. Explain the assumptions you are making about the probability model (if any).

Solution:

(a) Let X_A and X_B be the grade of a random student in section A and section B, respectively. The table tells us that $\mu_A = \mathrm{E}[X_A] = 65$ and $\mu_B = \mathrm{E}[X_B] = 70$. Markov's inequality only gives us a bound on the probability that X_A and X_B are *at least* as large as some value so they cannot be applied directly.

However, if we assume that the maximum grade is 100, then we can apply Markov's inequality to the nonnegative random variable $Y_A = 100 - X_A$:

$$\mathrm{P}(X_A \leq 30) = \mathrm{P}(Y_A \geq 70) \leq \frac{\mathrm{E}[Y_A]}{70} = \frac{35}{70} = \frac{1}{2}.$$

Similarly,

$$\mathrm{P}(X_B \leq 30) = \mathrm{P}(Y_B \geq 70) \leq \frac{\mathrm{E}[Y_B]}{70} = \frac{30}{70} = \frac{3}{7}.$$

Therefore we can say that the number of students with grade 30 or below is at most $30 \cdot 1/2 + 20 \cdot 3/7 \leq 23.6$, so there are at most 23 of them.

(b) From the table, $\sigma_A = \sqrt{\mathrm{Var}[X_A]} = 5,$, $\sigma_B = \sqrt{\mathrm{Var}[X_B]} = 10$. By Chebyshev's inequality, for a random student in section A,

$$\mathrm{P}(X_A \leq 30) = \mathrm{P}(X_A \leq \mu_A - 7 \cdot \sigma_A) \leq \mathrm{P}(|X_A - \mu_A| \geq 7\sigma_A) \leq 1/49 \approx 0.0204.$$

Although we are only interested in the probability that X_A is 7 standard deviations smaller than its mean, Chebyshev's inequality only tells us the probability of the possibly larger event that X_A is either 7 standard deviations smaller or 7 standard deviations larger than its mean. This is already a tremendously small probability – about 2%.

Similarly, for a random student in section B,

$$\mathrm{P}(X_B \leq 30) = \mathrm{P}(X_B \leq \mu_B - 4 \cdot \sigma_B) \leq \mathrm{P}(|X_B - \mu_B| \geq 4\sigma_B) \leq 1/16 \approx 0.00625.$$

Since there are 30 students in section A, at most $1/49 \cdot 30$ students must have received 30 or below, so nobody got that kind of grade. In section B, at most $1/16 \cdot 20$ students got 30 or below, so at most one student in the whole class could have received 30 or below on the exam.

(c) In order to apply the Central Limit Theorem we need to model the grade X of a random student as a sum of many independent random variables. The sample space here consists of 50 students, which is quite small to allow representing a grade as a sum of many independent random variables. So the Central Limit Theorem is not really applicable to this problem.

If we however pretended that the sample space was larger, the Central Limit Theorem would suggest modeling X_A as $\mu_A + N_A\sigma_A$ and X_B as $\mu_B + N_B\sigma_B$, where N_A and N_B are Normal$(0, 1)$ random variables. Then we would get that

$$\mathrm{P}(X_A \leq 30) \approx \mathrm{P}(N_A \leq (30 - \mu_A)/\sigma_A) = \mathrm{P}(N_A \leq -7) \approx 0$$

and

$$P(X_B \leq 30) \approx P(N_B \leq (30 - \mu_B)/\sigma_B) = P(N_B \leq -4) \approx 0,$$

leading to the conclusion that nobody got a grade below 30.

4. There are 6 computers. Every pair of computers connects with probability 10%, independently of the other pairs. Say a computer is *isolated* if it didn't connect to any of the other computers. Let N be the number of isolated computers.

   (a) Calculate the expected value of N.

   (b) Calculate the variance of N.

   (c) Argue that the probability that at least one computer is isolated is 70% or more.

   Solution: Let X_i be an indicator random variable for the event that computer i is isolated. Then $N = X_1 + \cdots + X_6$.

   (a) By linearity of expectation $E[N] = E[X_1] + \cdots + E[X_6]$. The value $E[X_1] = P(X_1 = 1)$ is the probability of the event that computer 1 is isolated. Because the computers connect independently with probability 0.1, the number of connections computer 1 makes is a Binomial$(5, 0.1)$ random variable. Computer 1 is isolated when it makes zero connections, which happens with probability $(0.9)^5$, so $E[X_1] = (0.9)^5$. Similarly $E[X_2] = \cdots = E[X_6] = (0.9)^5$ and so $E[N] = 6 \times (0.9)^5 \approx 3.543$.

   (b) The variance of N can be computed using the following formula:

   $$\mathrm{Var}[N] = \mathrm{Var}[X_1 + \cdots + X_6] = \sum_{i=1}^{6} \mathrm{Var}[X_i] + \sum_{i \neq j} \mathrm{Cov}[X_i, X_j].$$

   Like before $\mathrm{Var}[X_i] = P(X_i = 1)(1 - P(X_i = 1)) = (0.9)^5(1 - (0.9)^5)$. Similarly $\mathrm{Cov}[X_i, X_j] = P(X_i = 1, X_j = 1) - P(X_i = 1)\,P(X_j = 1)$ and $P(X_i = 1) = P(X_j = 1) = (0.9)^5$. It remains to calculate $P(X_i = 1, X_j = 1)$. The event "$X_i = 1$ and $X_j = 1$" occurs when both computers i and j are isolated. This happens when all 9 connections that these two computers are involved in fail to happen. Since these connections are independent and each has probability 0.1, $P(X_i = 1, X_j = 1) = (0.9)^9$ so we get that

   $$\mathrm{Cov}[X_i, X_j] = (0.9)^9 - (0.9)^5(0.9)^5 = (0.9)^9 - (0.9)^{10} = 0.1 \times 0.9^9.$$

   Therefore the variance of N is given by:

   $$\mathrm{Var}[N] = \sum_{i=1}^{6} Var[X_i] + \sum_{i \neq j} \mathrm{Cov}[X_i, X_j] = 6 \times 0.9^5(1 - 0.9^5) + 6 \times 5 \times (0.1 \times 0.9^9) \approx 2.613.$$

   (c) The expectation of N is $\mu \approx 3.543$ and its standard deviation is $\sigma \approx \sqrt{2.613} \approx 1.616$. By Chebyshev's inequality, we have:

   $$P(N = 0) = P(N \leq 0) \leq P(|N - \mu| \geq 2.192\sigma) \leq 1/(2.192)^2 \leq 0.208 < 30\%$$

   and so P(at least one computer is isolated) $= P(N > 0) = 1 - P(N = 0) > 70\%$.